A Fast and Deterministic Method for Mean Time to Fixation in Evolutionary Graphs

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Abstract

Title: A Fast and Deterministic Method for Mean Time to Fixation in Evolutionary Graphs

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Abstract: The propagation of a given phenomena in a social network is an important topic of research in network science. This presentation introduces our study of the "mean time to fixation" (MTF) problem – that is the average amount of time required for a phenomenon to spread to the entire population in the network. We study this problem under the well-known propagation models of the invasion process, voter dynamics, and link dynamics. We devise an algorithm that produces a provable and non-trivial lower bound on MTF as well as show this approach to be experimentally viable. Our results provide insight into various propagation problems on social networks -– including the non-monotonic spread of influence and emergence of cooperation.

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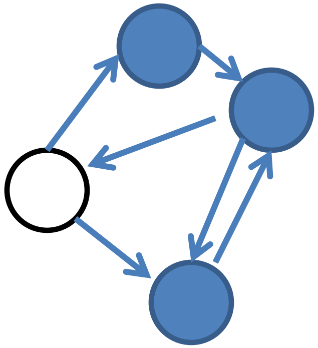
Evolutionary Graphs in Social Sciences

Take a population of individuals who may either choose to cooperate or defect with a neighbor in any interaction. The relationships between these individuals may be modeled with an evolutionary graph.

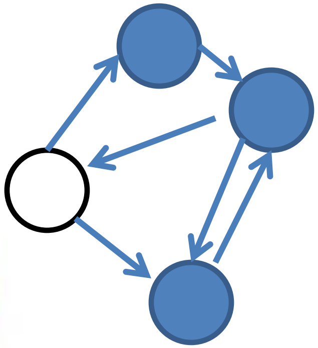
The time it takes for cooperation to spread throughout the population is the mean time to fixation.

Network Diffusion

Monotonic



Non-Monotonic



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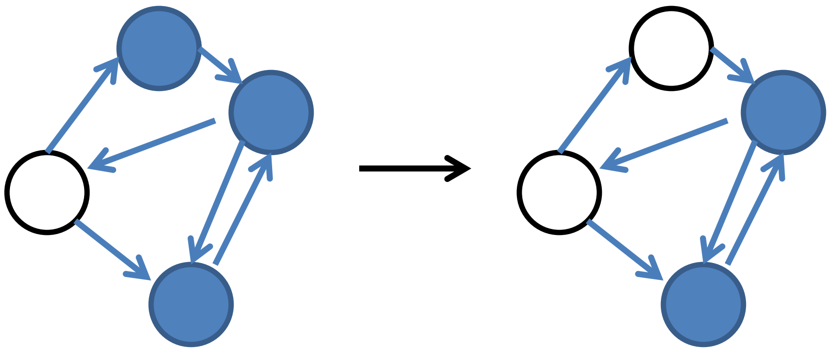
Network Diffusion

Monotonic



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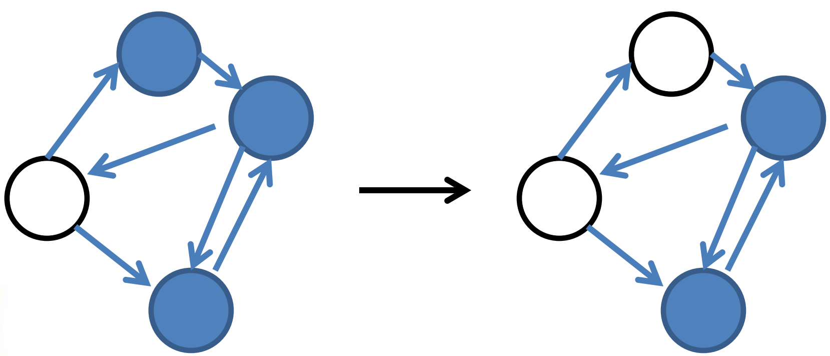

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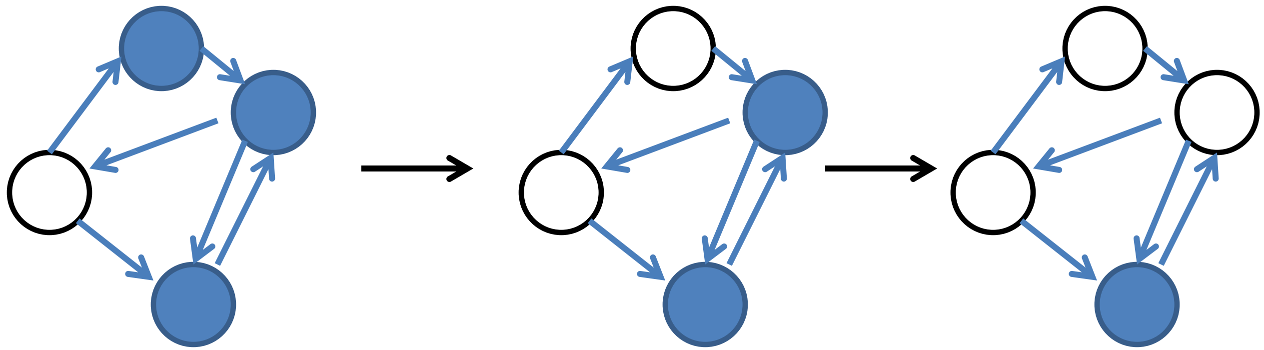
Network Diffusion

Monotonic



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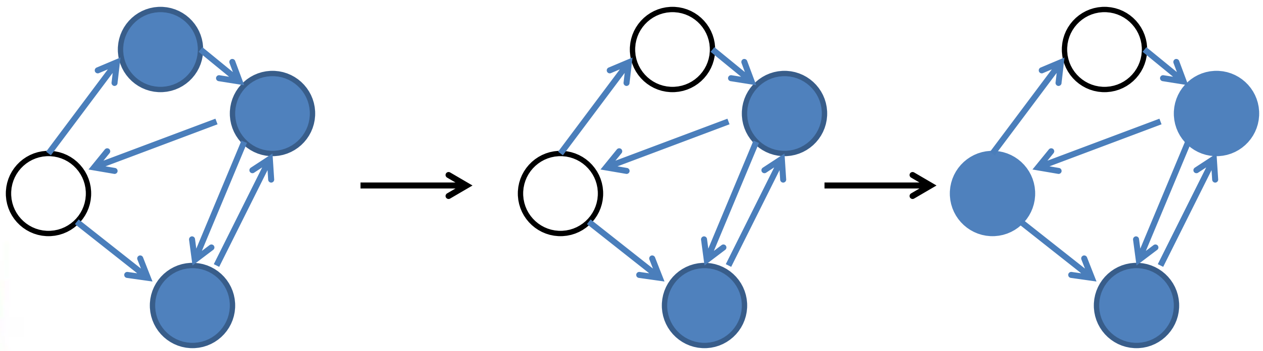

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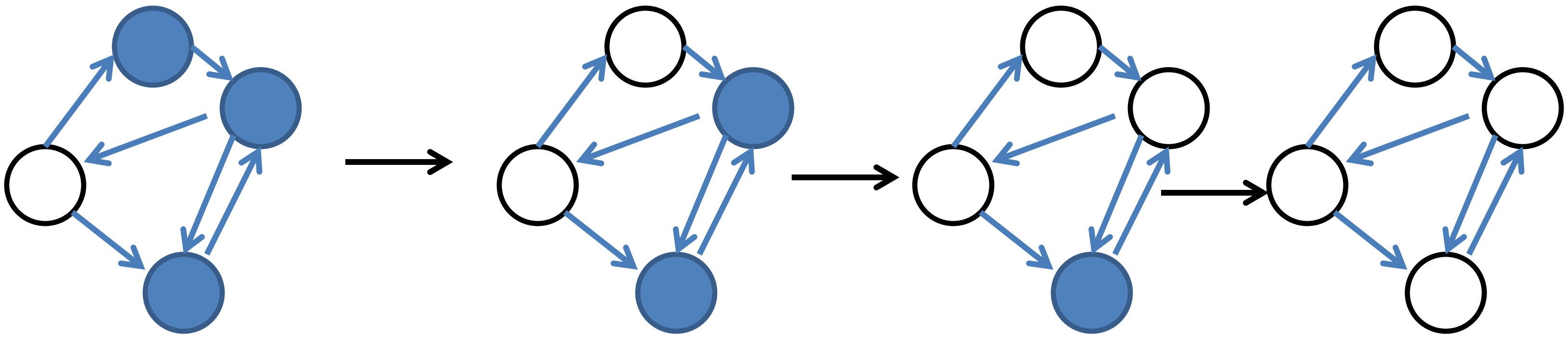
Network Diffusion

Monotonic



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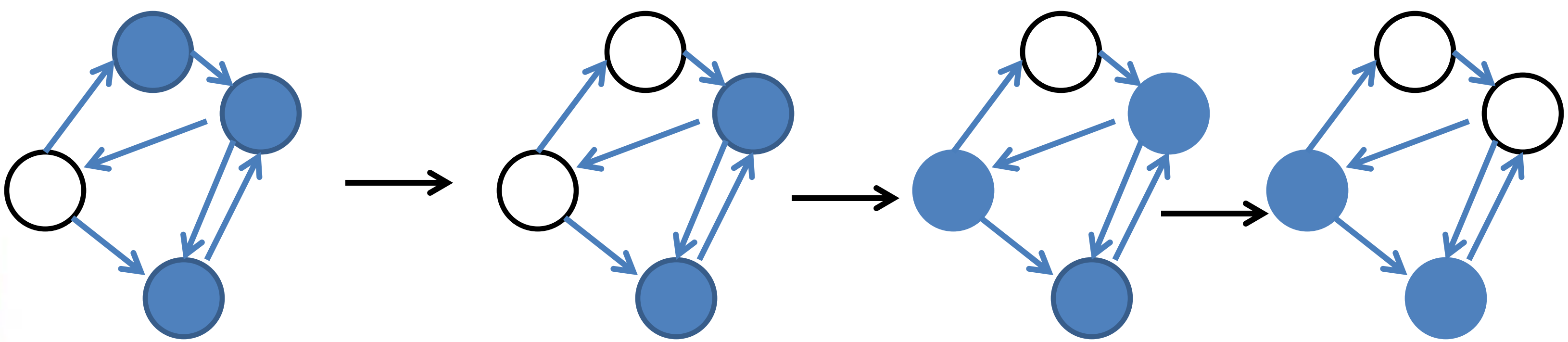


Non-Monotonic Diffusion

Non-monotonic graphs are studied in evolutionary graph theory, where a population is modeled as a network of **mutant** and **resident** nodes. [1]

Every time step the number of **mutants** may either increase or decrease.

Non-Monotonic Diffusion: The Birth-Death Process

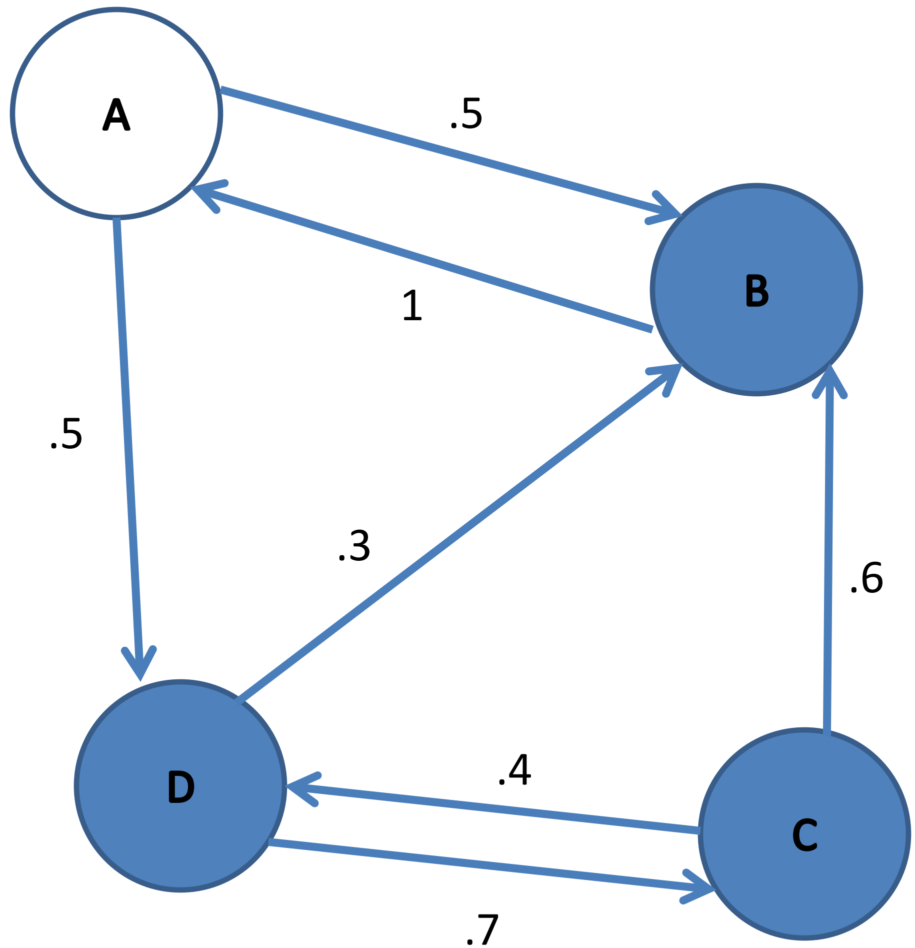
This research was done with a **birth-death process**.

At each time step, a **node B** is chosen **without preference** from the graph.

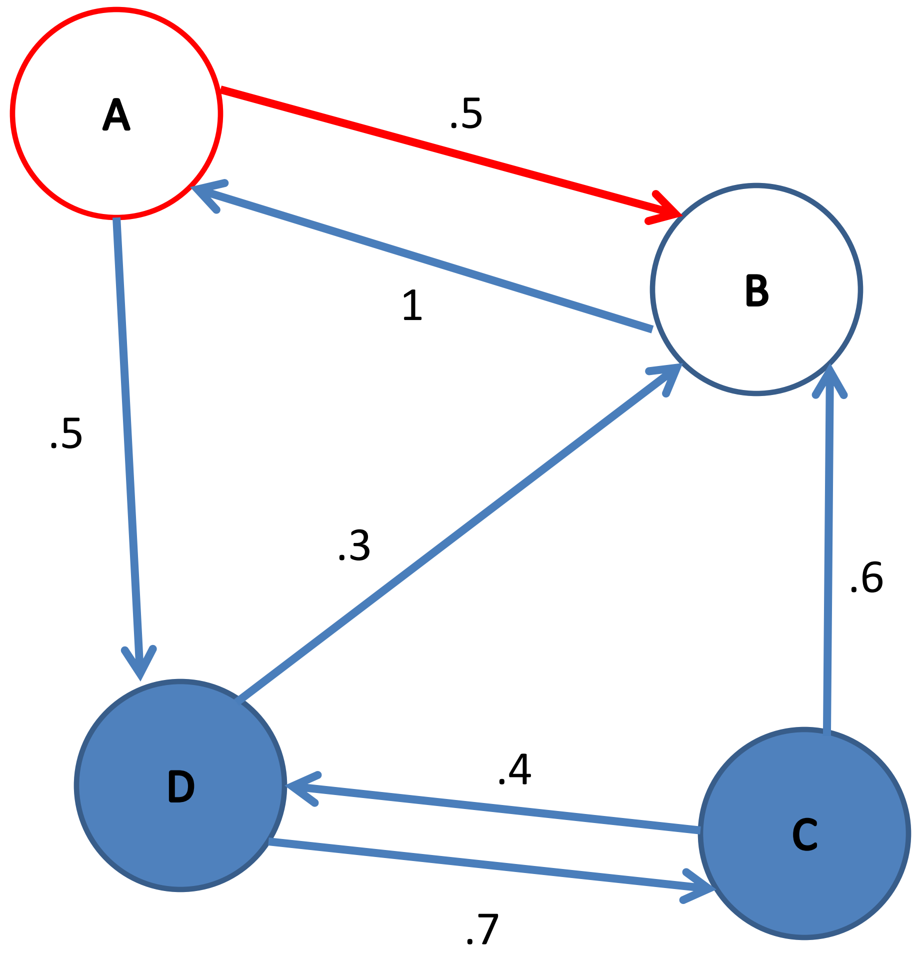
D is chosen with a probability **proportionate** to the **weight of outgoing edges** from **B**.

B's character, **resident or mutant**, is cloned onto **D**.

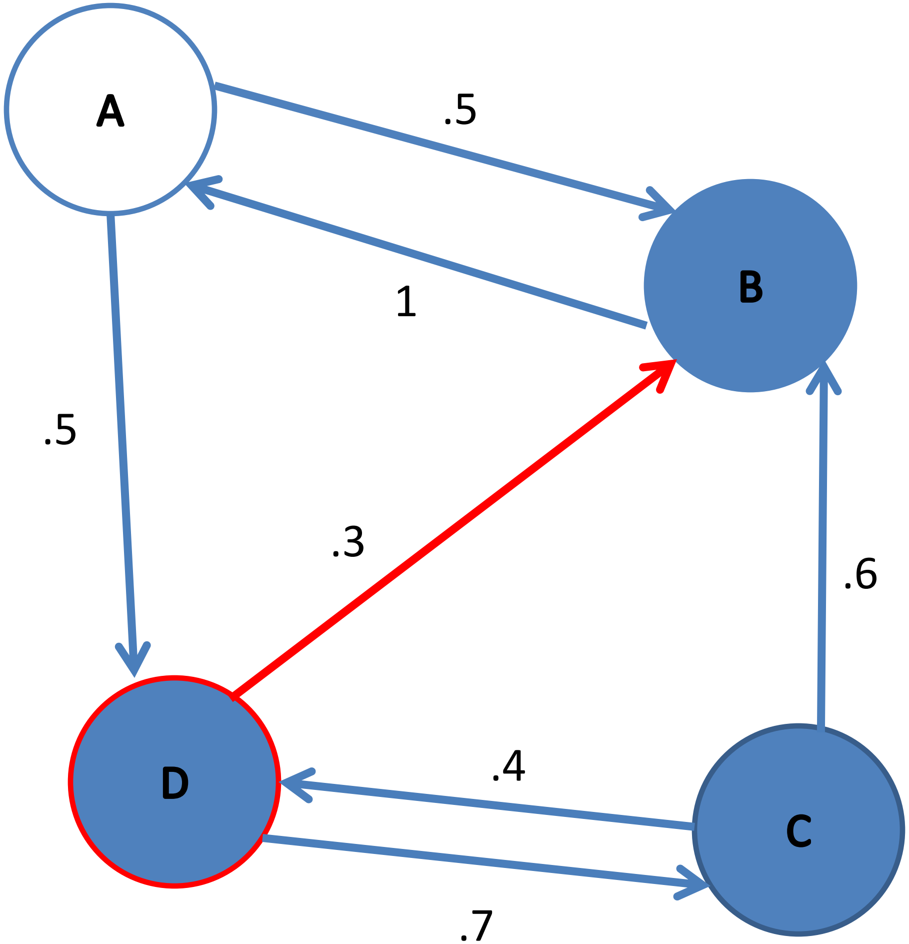
Birth-Death Process



Birth-Death Process



Birth-Death Process



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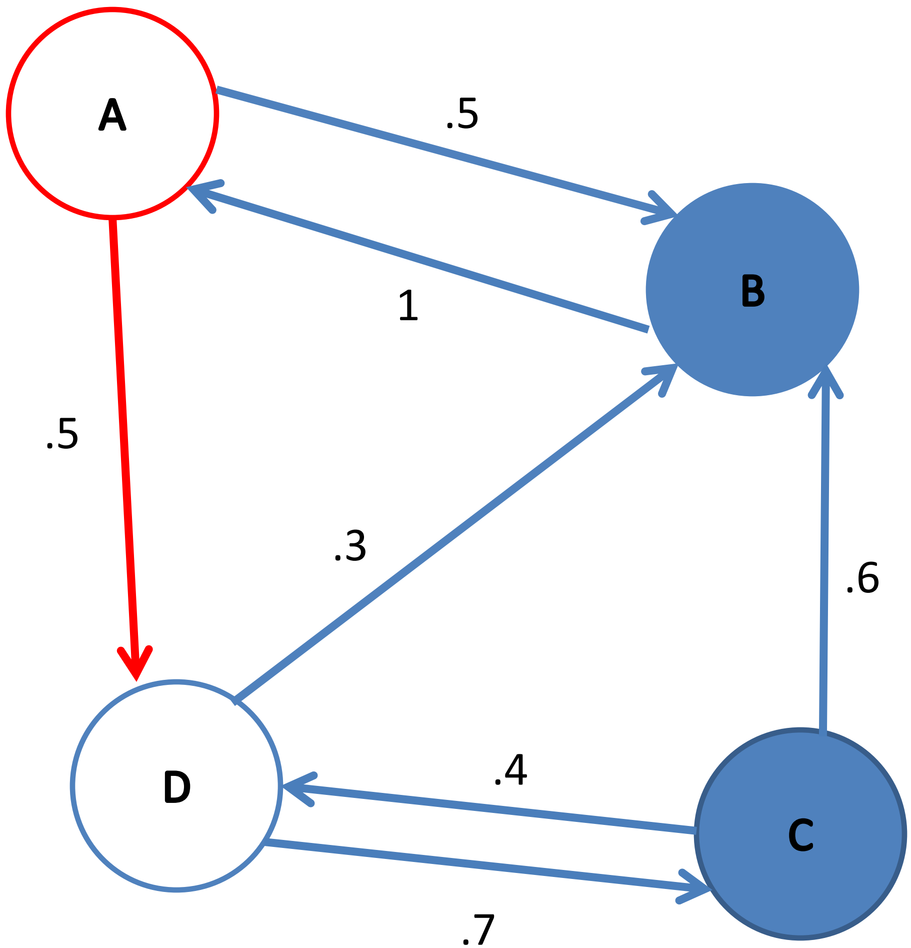
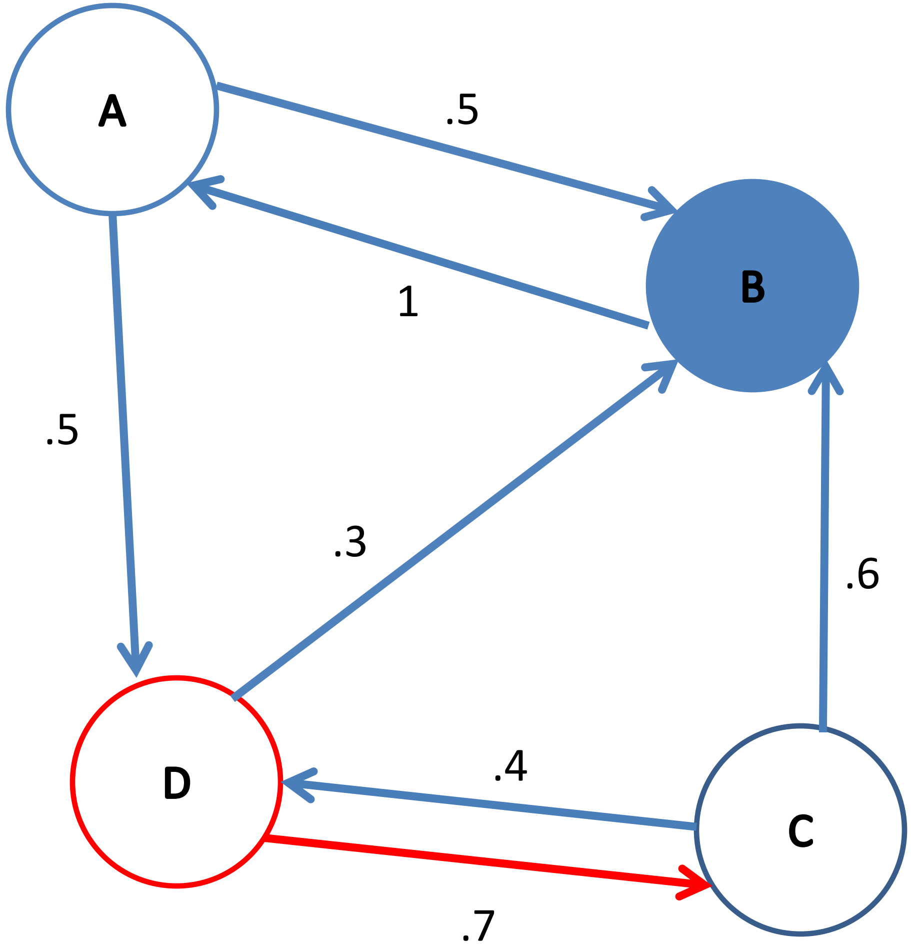
Birth-Death Process



Birth-Death Process



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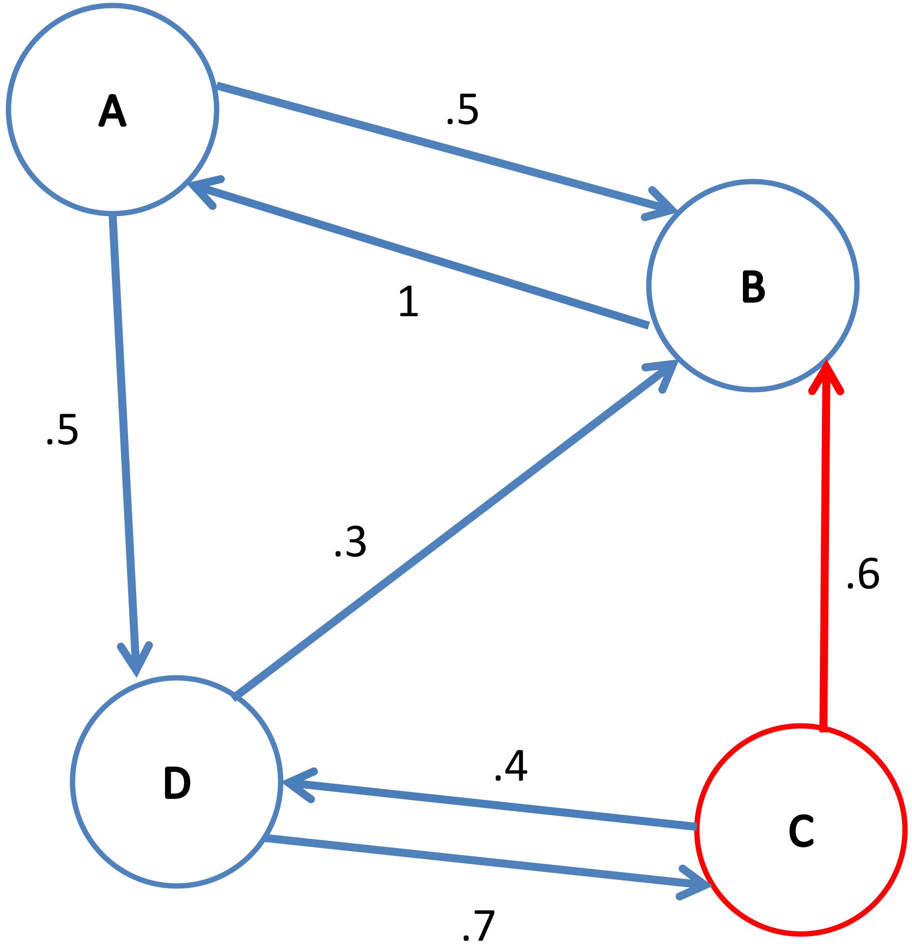
Birth-Death Process



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Non-Monotonic Diffusion: Death-Birth and Link-Dynamics

Death-Birth: An alternative method where a **node is chosen impartially to die**, and a neighbor node is chosen proportional to the **incoming edges** and **cloned over the dead node**.

Link-Dynamics: **Edges** are chosen, and transfer of mutant or resident properties moves across the **edge**.

Non-Monotonic Diffusion

Other variants on the aforementioned models exist which have undirected edges.

The link dynamics, death birth and other variant processes **were not used** in this research, but **our theorem can be applied** to them as well.

Applications of Evolutionary Graphs

Evolution of Cooperation in Social Networks
- [2] Ohtsuki, Hauert, Lieberman, & Nowak

Animal Migrations
- [6] Zhang, Nie, & Hu

Primate Habits
- [5] Voelkl & Kasper

Interactive Particle Systems
- [4] Sood, Antal, & Redner

Fixation Probability

The **probability** that a graph with an initial configuration of mutants and residents will result in a **completely mutant graph**.

Mean Time to Fixation

The **average time** it takes for a graph **to reach fixation.**

Fixation Probability Approaches

- Monte Carlo simulations
- Shortened Simulation
  - assume fixation once a certain mutant density threshold is reached
- Special Case analyzed for particular solutions
- Deterministic Algorithms

Mean Time to Fixation Approaches

Monte Carlo Simulations

Special Cases analyzed for particular solutions

Mean Time to Fixation Approaches

Monte Carlo Simulations

Special Cases analyzed for particular solutions

Can we develop a deterministic method for mean time to fixation?

Mean Time to Fixation

$$t_C = \frac{1}{P_C} \sum_{t=1}^{\infty} t \cdot (P_C^{(t)} - P_C^{(t-1)})$$

Time to fixation, t_c, traditionally relies on the probability of fixation P_c at each time step.

Theorem

Shakarian and Roos [2] developed an algorithm which provides probability of being a mutant at each time step for each vertex. They prove that vertex probabilities and fixation probability converge as time goes to infinity.

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Theorem

$$t_c \geq \frac{1}{\mathrm{P_c}} \sum_{t=1}^{\infty} t\left[min_i(Pr_i^t) - min_i\left(Pr_i^{t-1}\right)\right]$$

$min_i(Pr_i^t)$ is the minimum probability of being a mutant for any node in the graph at time *t.*

As $min_i(Pr_i^t)$ is an upper bound on P_C^t, we use an accounting method to prove the bound above.

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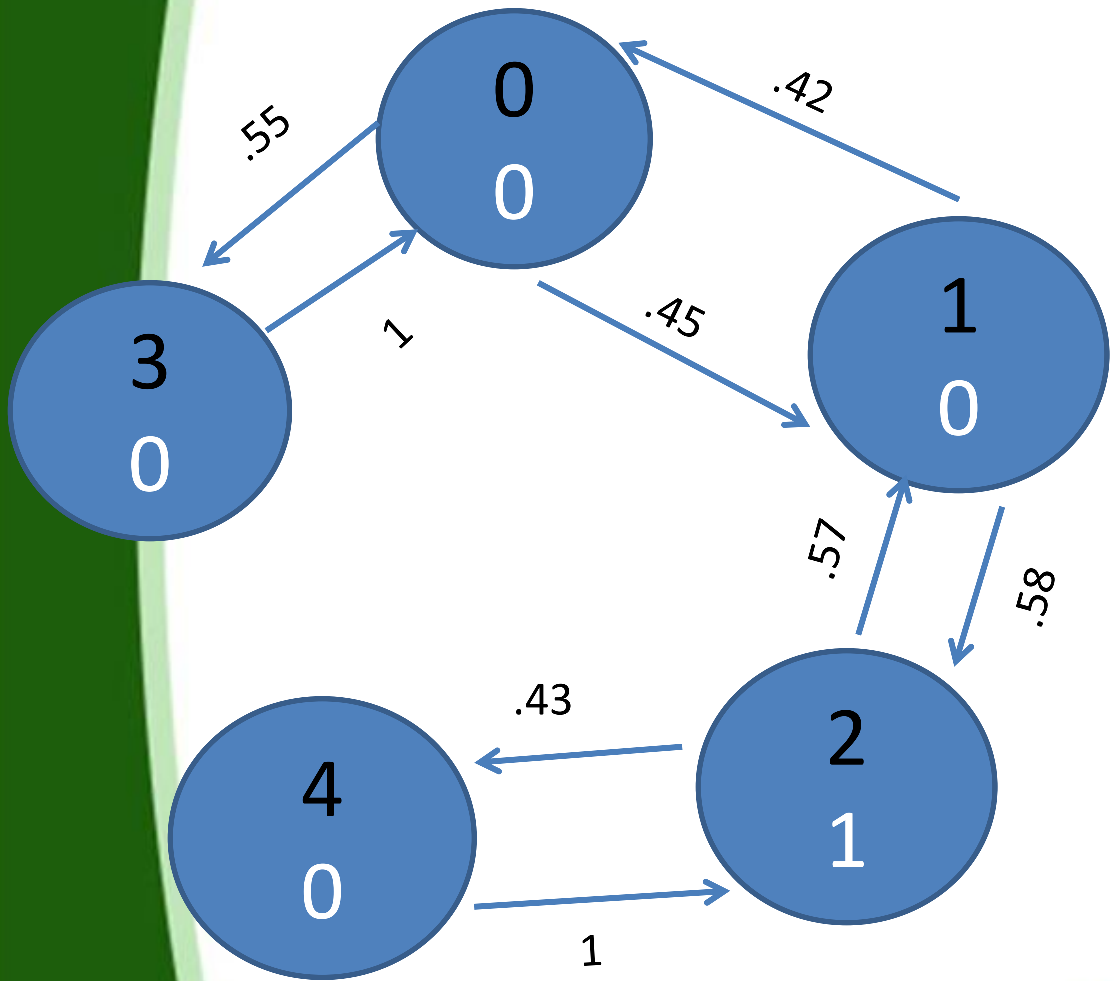
Demonstration

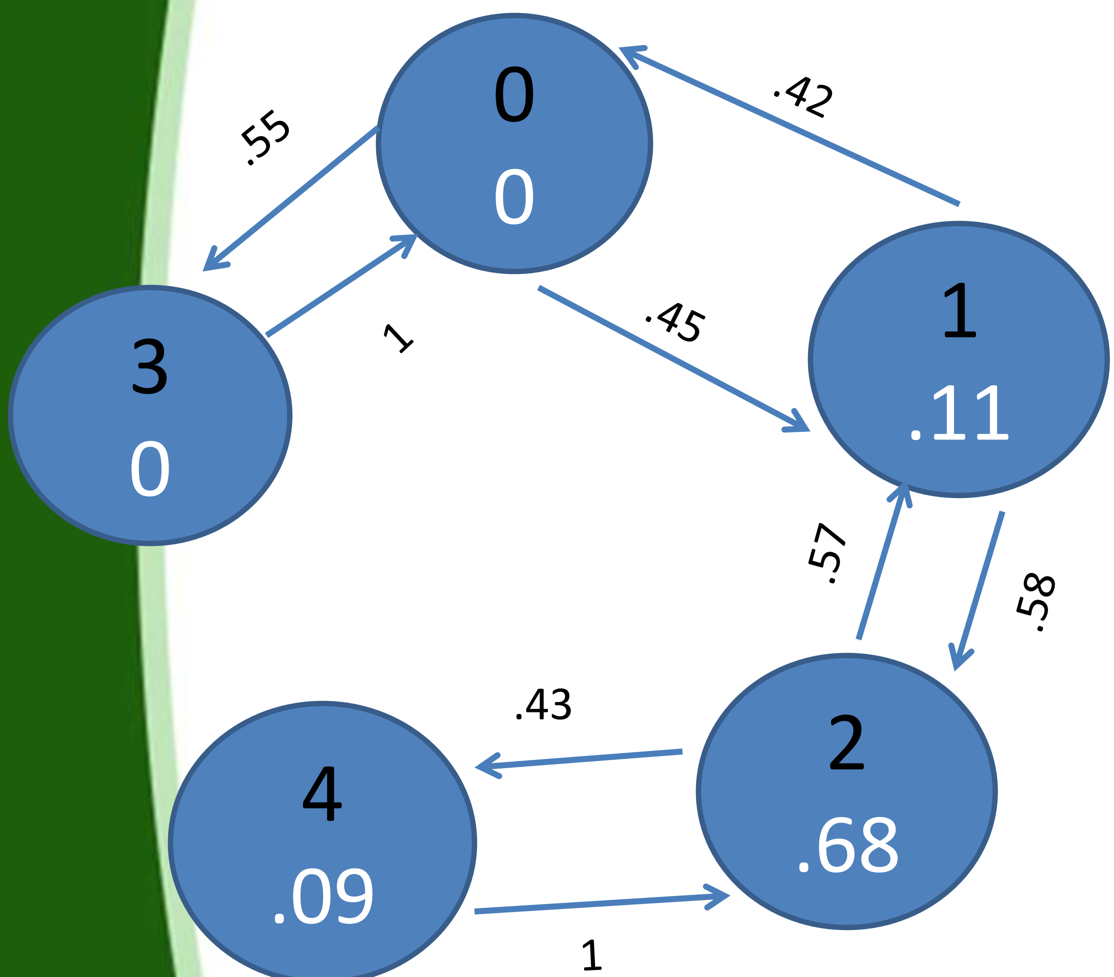


Nodes are circles with:
Black: Node Index
White: Pr_i^t

Arrows indicate edges and are marked with their weights.

Demonstration

$$t_c \geq \frac{1}{\mathrm{P_c}} \sum_{t=1}^{\infty} t\left[min_i(Pr_i^t) - min_i(Pr_i^{t-1})\right]$$



Time: **1**

Minimum *(Pr_i^t)*: **0**

Previous Min: **0**

Addition at t: **0**

Summation: **0**

Estimated Mean Time to Fixation: **0**

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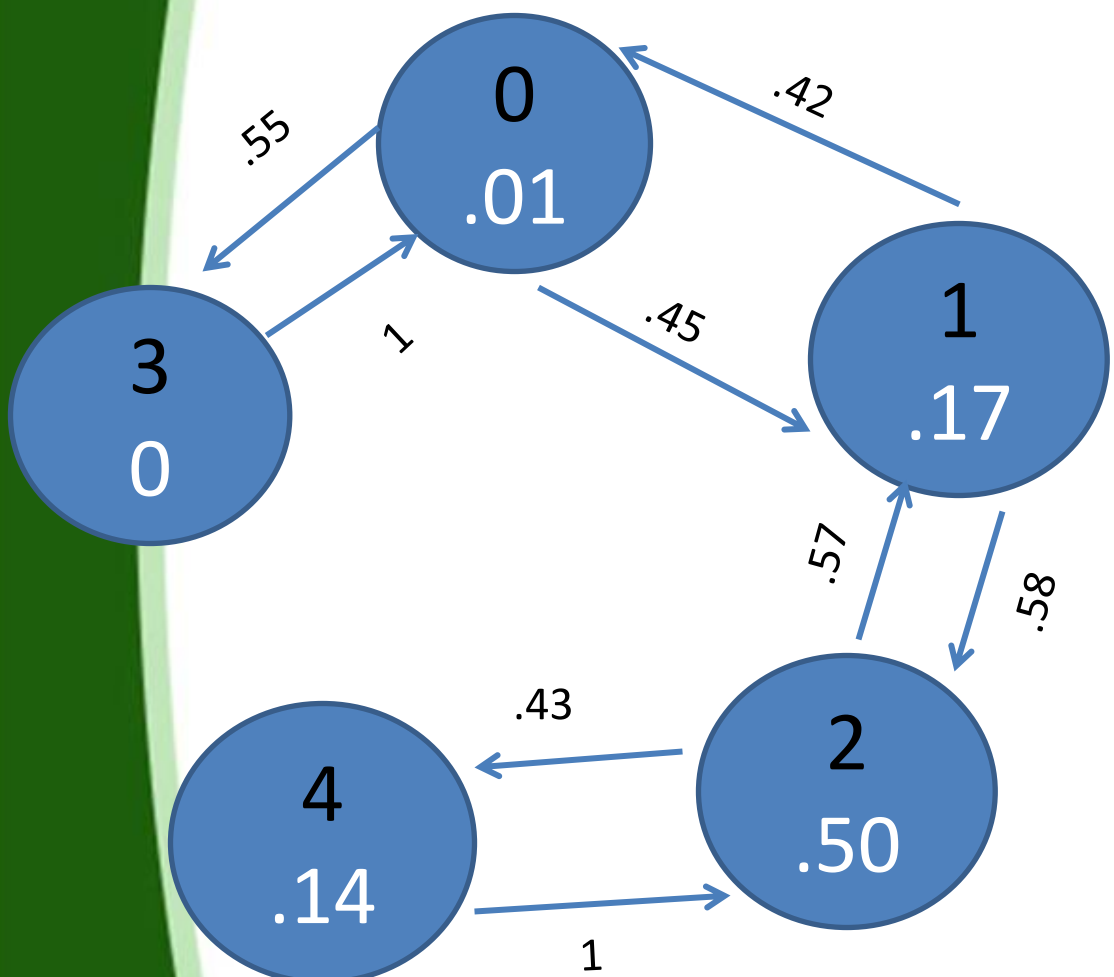
Demonstration

$$t_c \geq \frac{1}{\mathrm{P_c}} \sum_{t=1}^{\infty} t\left[min_i(Pr_i^t) - min_i(Pr_i^{t-1})\right]$$



Time: **2**

Minimum *(Pr_i^t)*: **0**
Previous Min: **0**

Addition at t: **0**

Summation: **0**
Estimated Mean Time to Fixation: **0**

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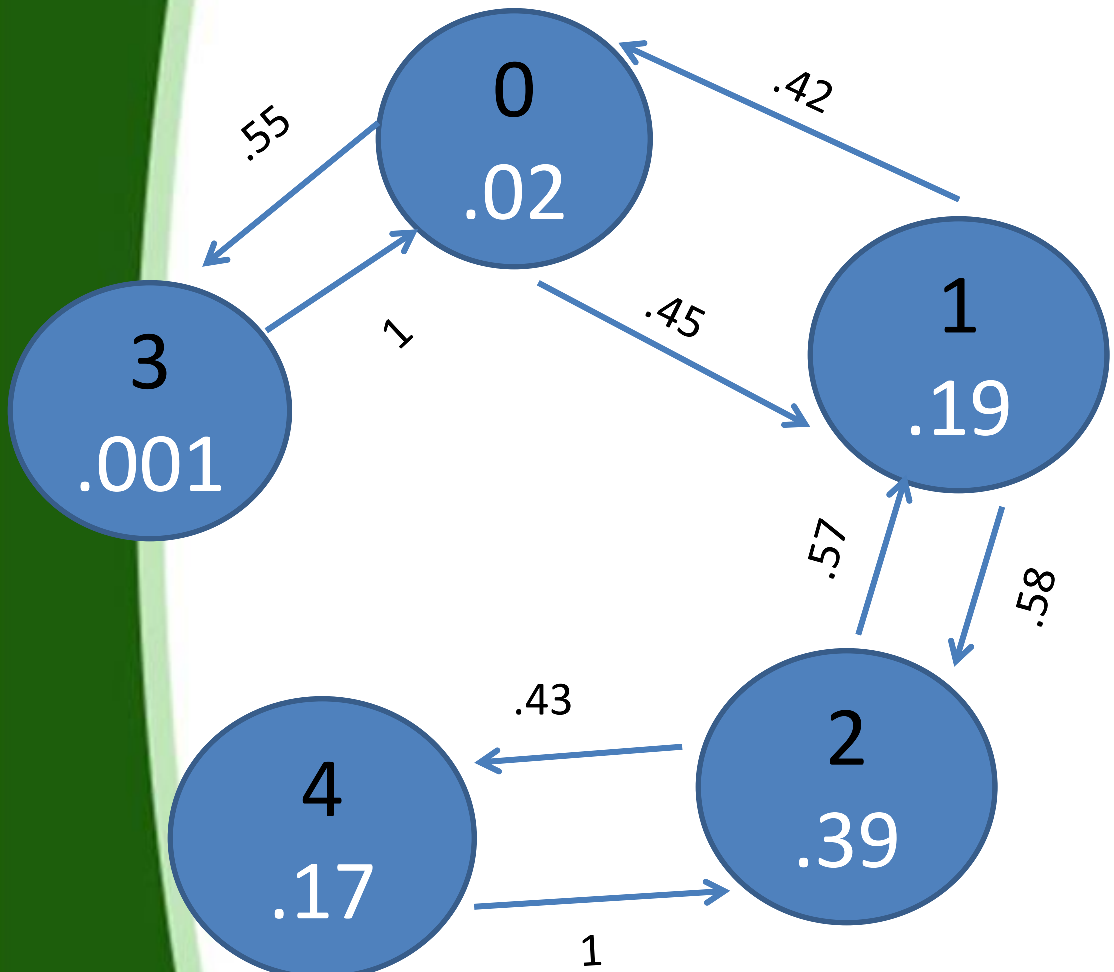
Demonstration

$$t_c \geq \frac{1}{\mathrm{P_c}} \sum_{t=1}^{\infty} t\left[min_i\left(Pr_i^t\right) - min_i\left(Pr_i^{t-1}\right)\right]$$



Time: **3**

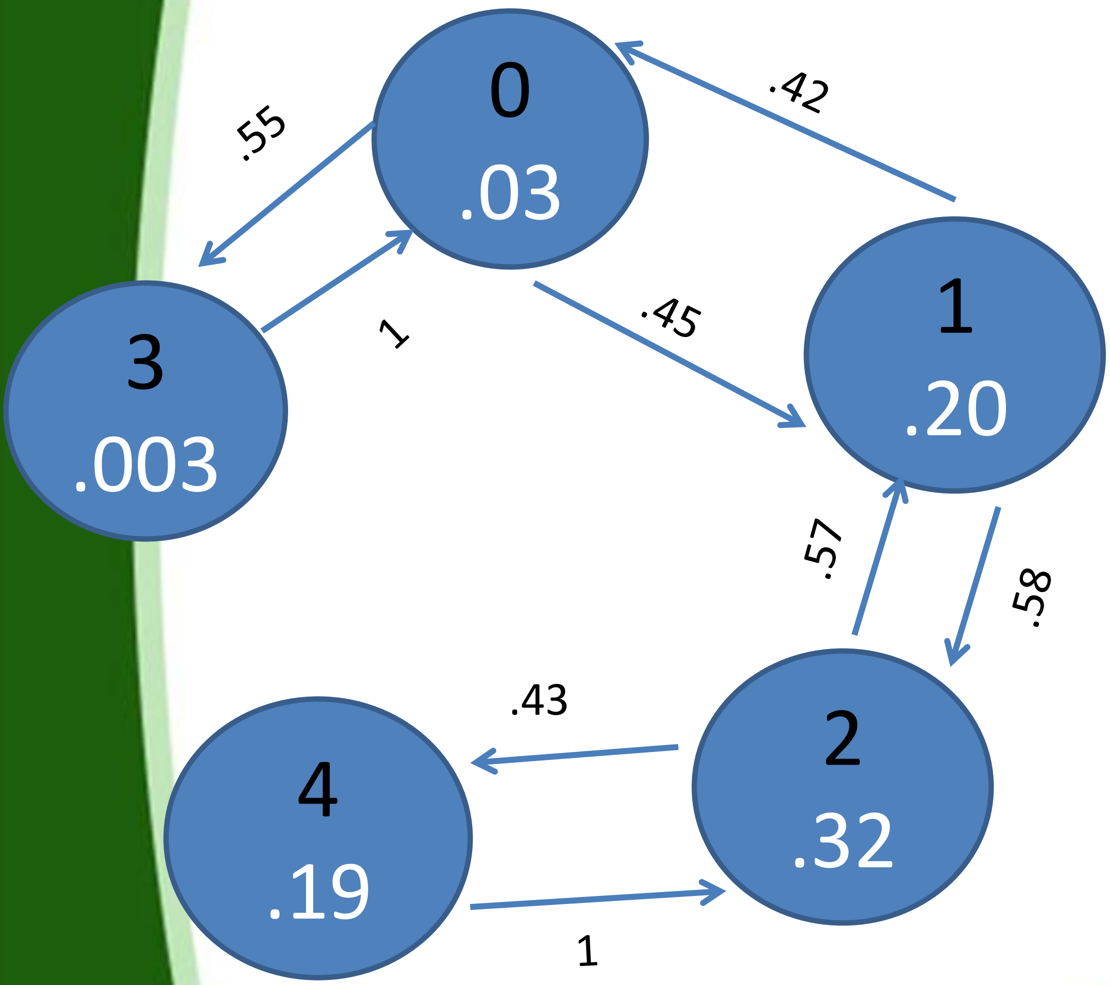
Minimum *(Pr_i^t)*: **.001**
Previous Min: **0**

Addition at t: **.003**

Summation: **.003**
Estimated Mean Time to Fixation: **.02**

Demonstration

$$t_c \geq \frac{1}{\mathrm{P_c}} \sum_{t=1}^{\infty} t\left[min_i\left(Pr_i^t\right) - min_i\left(Pr_i^{t-1}\right)\right]$$



Time: **4**

Minimum *(Pr_i^t)*: **.003**
Previous Min: **.001**

Addition at t: **.008**

Summation: **.012**
Estimated Mean Time to Fixation: **.08**

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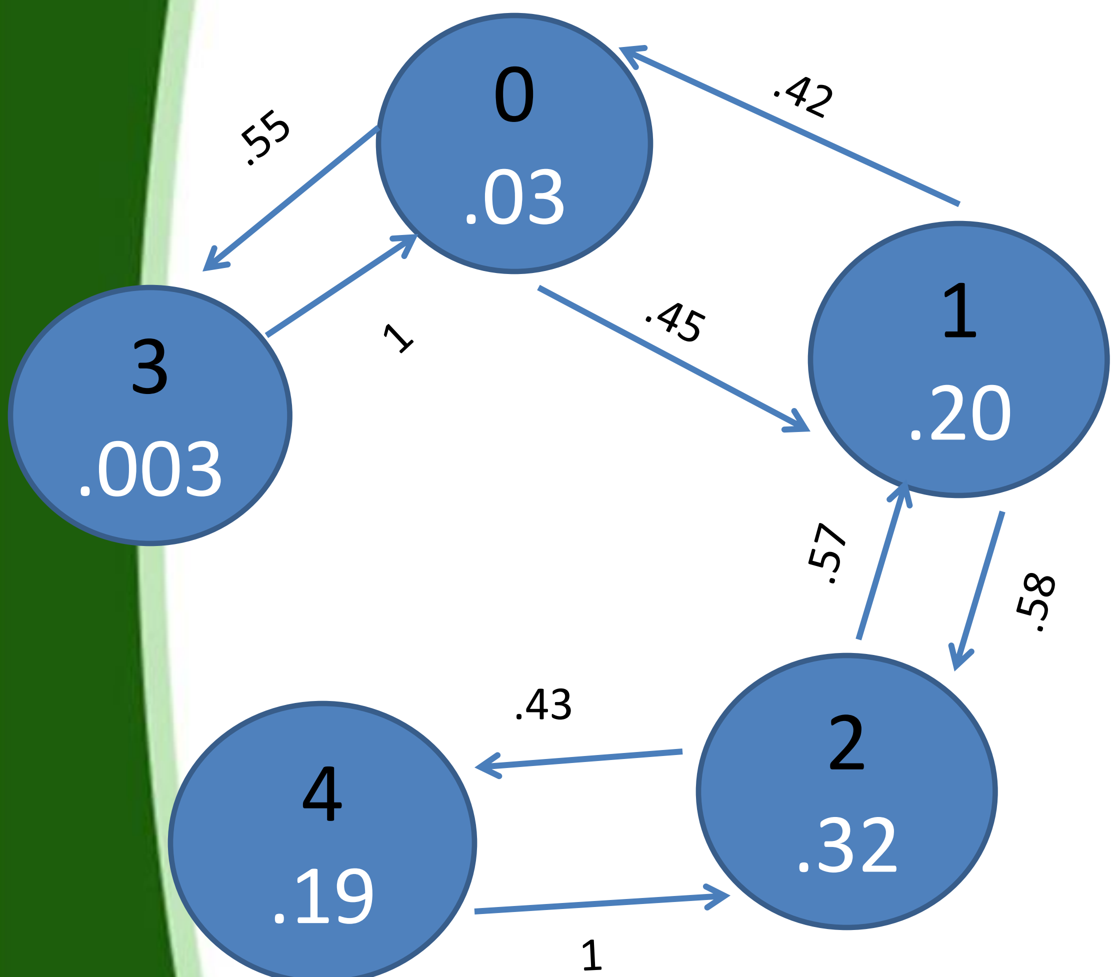
Demonstration

$$t_c \geq \frac{1}{P_c}\sum_{t=1}^{\infty} t\left[min_i(Pr_i^t) - min_i(Pr_i^{t-1})\right]$$



Time: **5**

Minimum *(Pr_i^t)*: **.006**

Previous Min: **.003**

Addition at t: **.015**

Summation: **.027**

Estimated Mean Time to Fixation: **.19**

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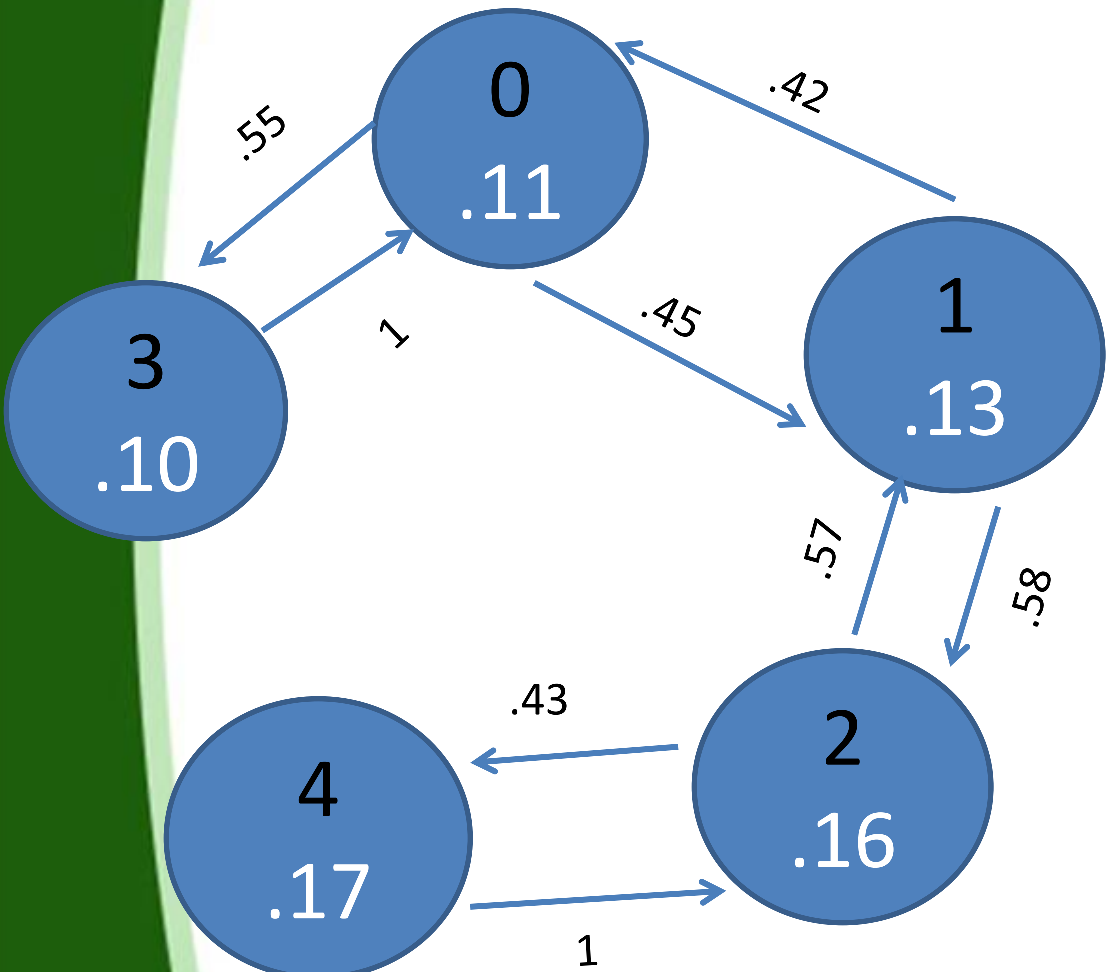
Demonstration

$$t_c \geq \frac{1}{P_c} \sum_{t=1}^{\infty} t\left[min_i(Pr_i^t) - min_i\left(Pr_i^{t-1}\right)\right]$$



Time: **50**

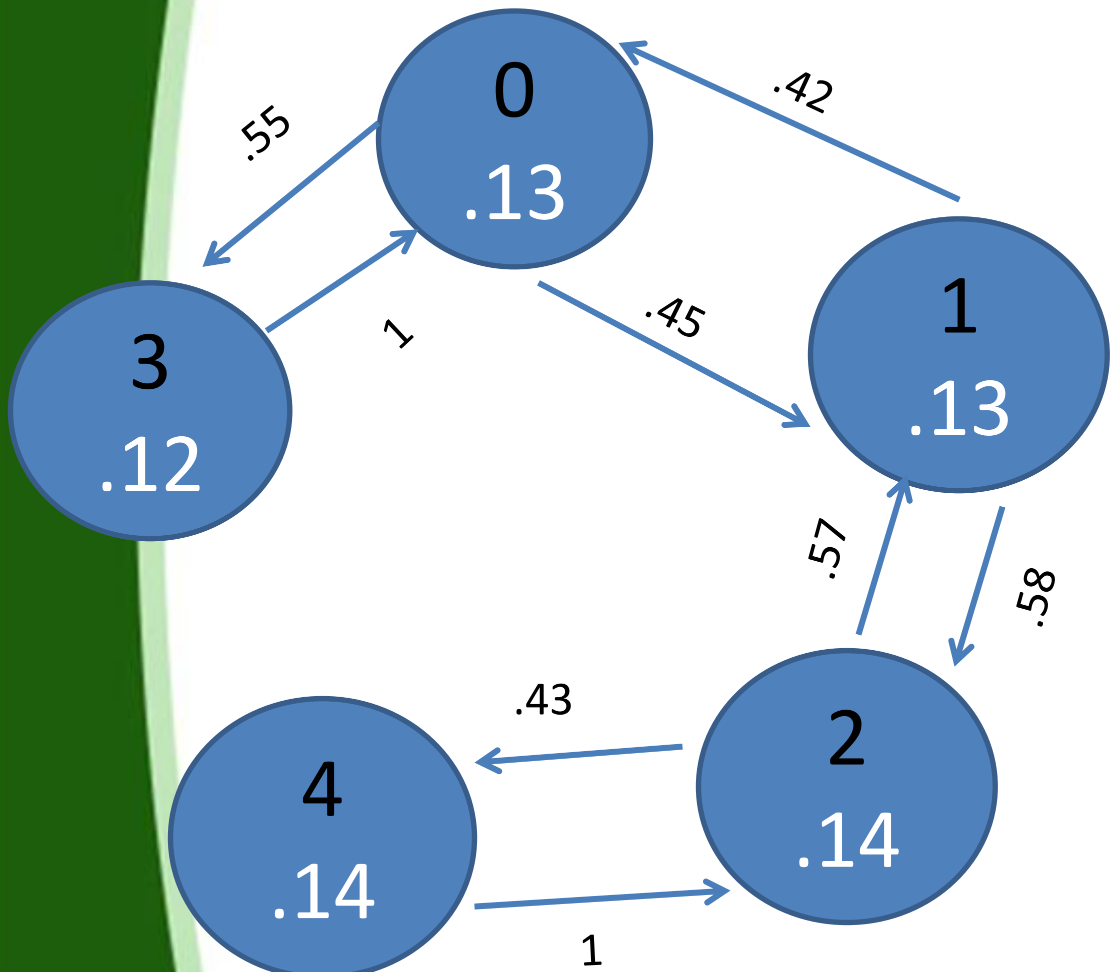
Minimum *(Pr_i^t)*: **.1**

Addition at t: **.046**

Estimated Mean Time to Fixation: **15.56**

Demonstration

$$t_c \geq \frac{1}{P_c} \sum_{t=1}^{\infty} t\left[min_i(Pr_i^t) - min_i\left(Pr_i^{t-1}\right)\right]$$



Time: **100**

Minimum *(Pr_i^t)*: **.12**

Addition at t: **.02**

Estimated Mean Time to Fixation: **28.79**

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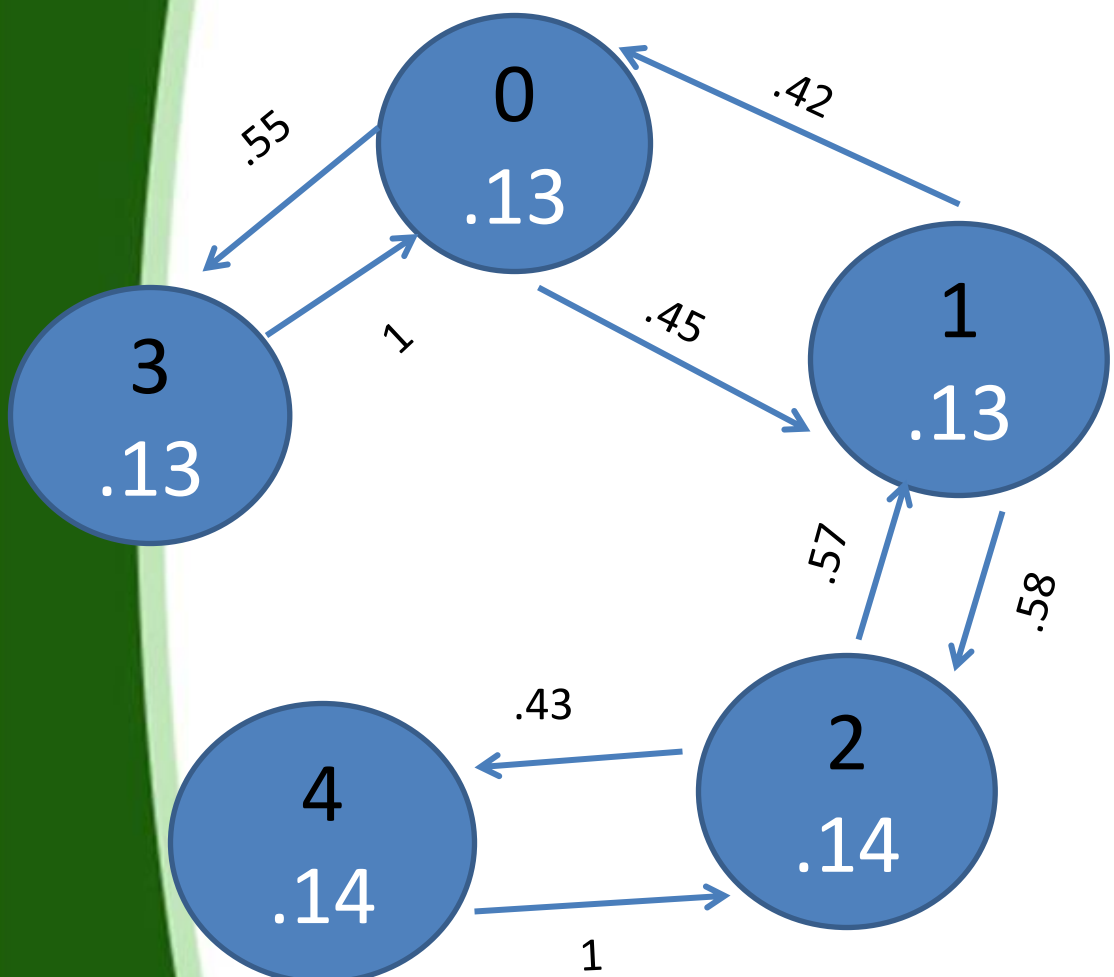
Demonstration

$$t_c \geq \frac{1}{\mathrm{P_c}} \sum_{t=1}^{\infty} t\left[min_i(Pr_i^t) - min_i(Pr_i^{t-1})\right]$$



Time: **150**

Minimum *(Pr_i^t):* **.13**

Addition at t: **.01**

Estimated Mean Time to Fixation: **34.96**

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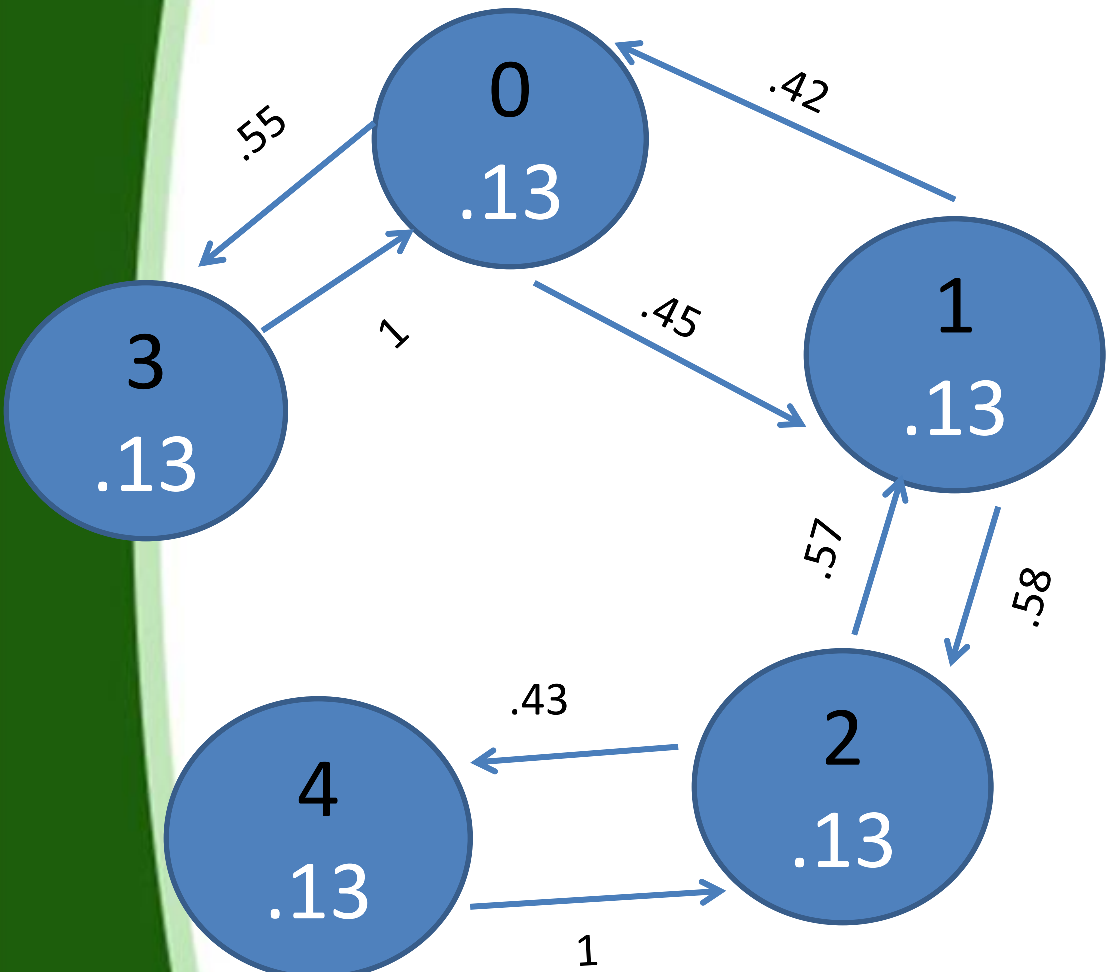
Demonstration

$$t_c \geq \frac{1}{\mathrm{P_c}} \sum_{t=1}^{\infty} t\left[min_i(Pr_i^t) - min_i\left(Pr_i^{t-1}\right)\right]$$



Time: **200**

Minimum *(Pr_i^t)*: **.13**

Addition at t: **.004**

Estimated Mean Time to Fixation: **37.34**

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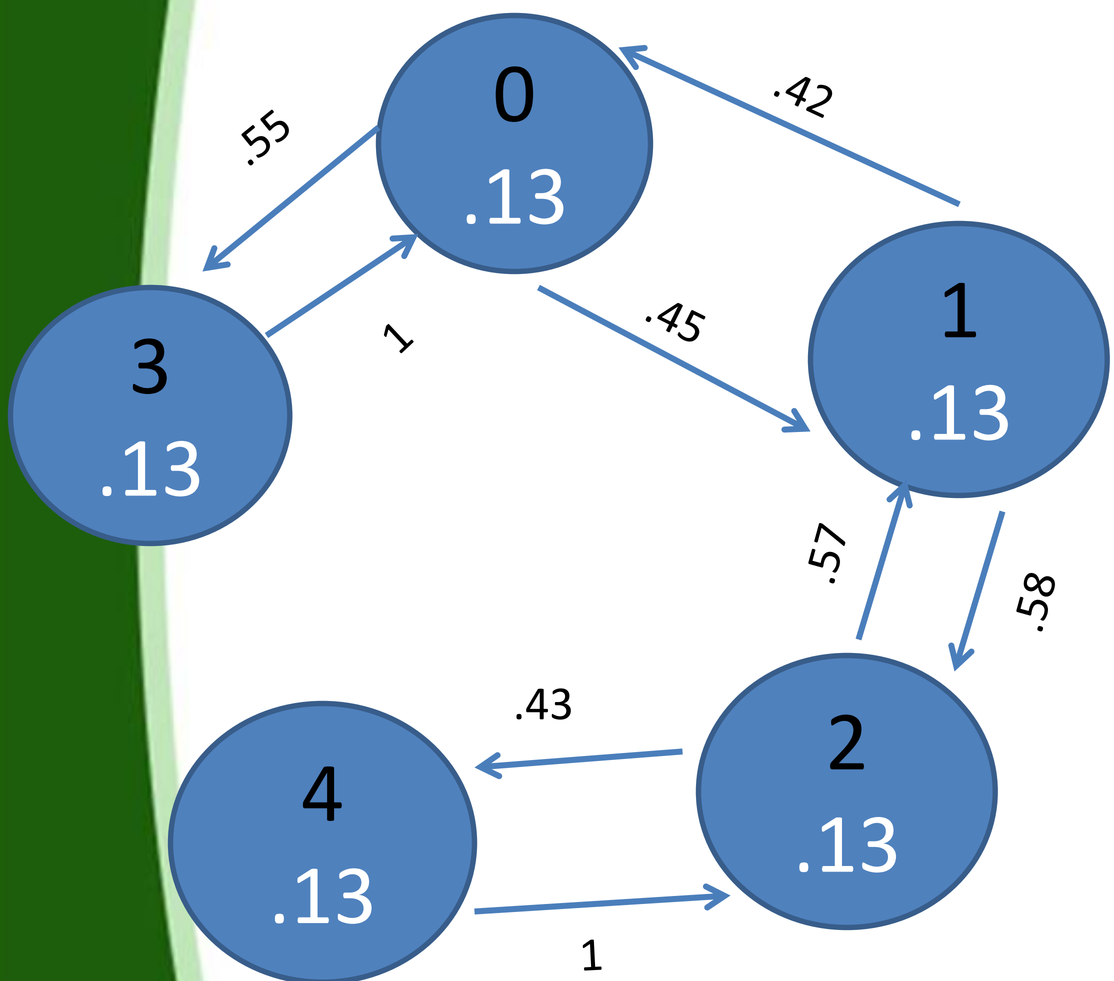
Demonstration

$$t_c \geq \frac{1}{\mathrm{P_c}} \sum_{t=1}^{\infty} t\left[min_i(Pr_i^t) - min_i\left(Pr_i^{t-1}\right)\right]$$



Time: **250**

Minimum *(Pr_i^t)*: **.13**

Addition at t: **.001**

Estimated Mean Time to Fixation: **38.18**

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Example Run

Simulation Mean Time to Fixation: 37 steps

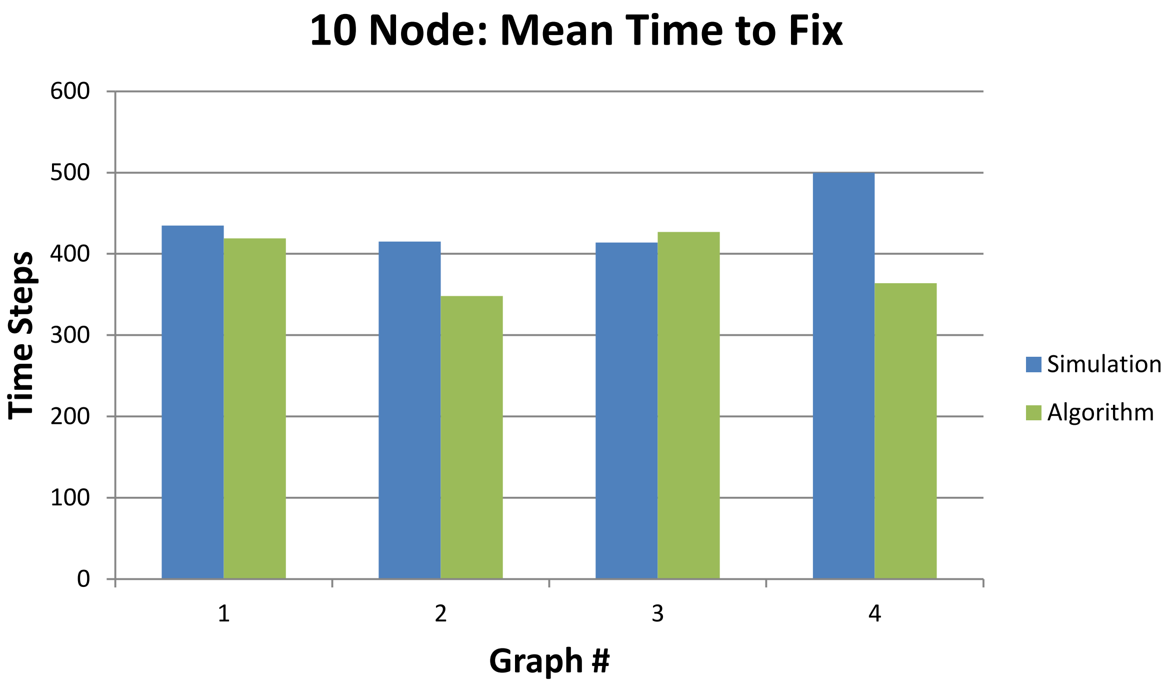
averaged over 10,000 trials in .946 sec.

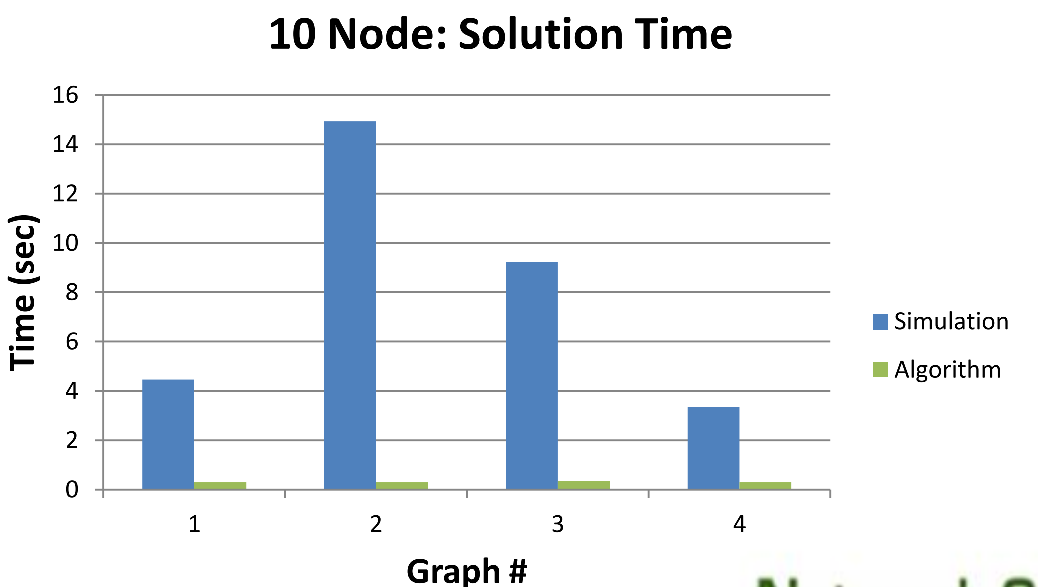
Algorithm Mean Time to Fixation: 38 steps

Convergence at $STD[Pr(M^{(t)}_i)] <= .00001$ in .031 sec.

Results

The following represent a sampling of the scale free graphs in our preliminary tests.





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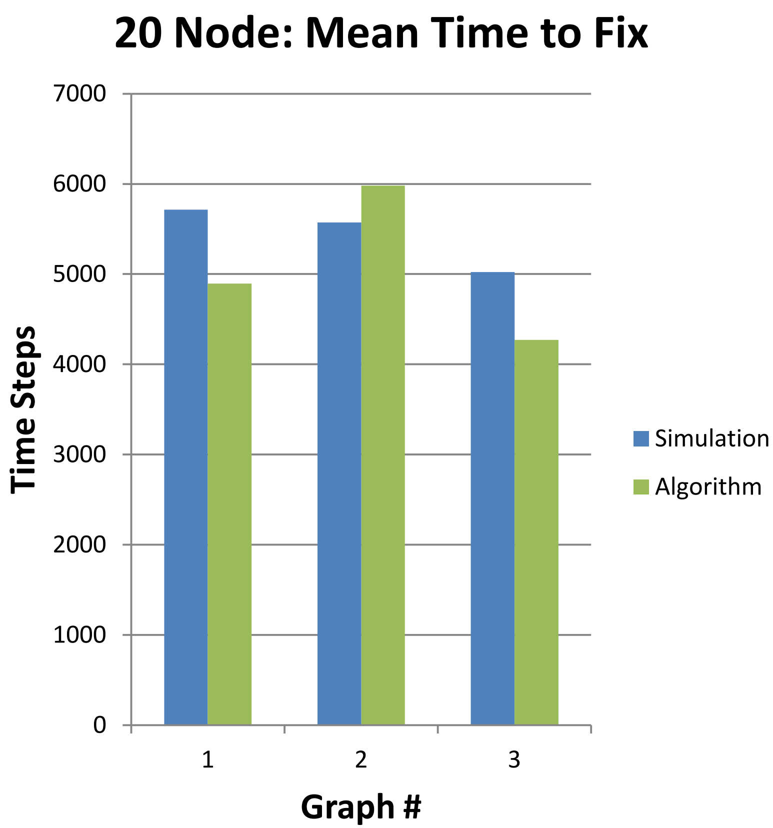

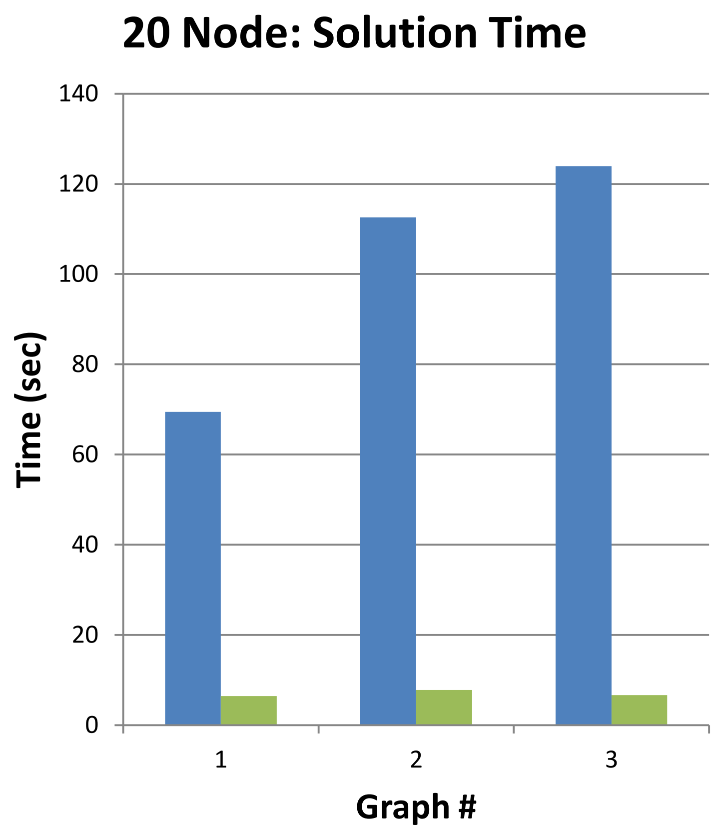

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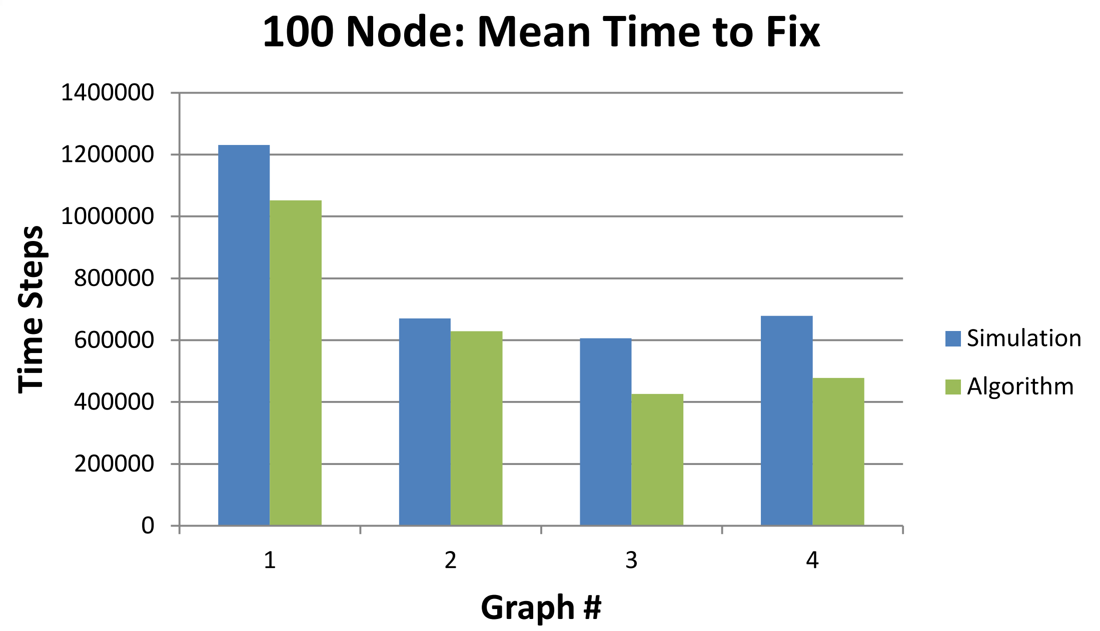

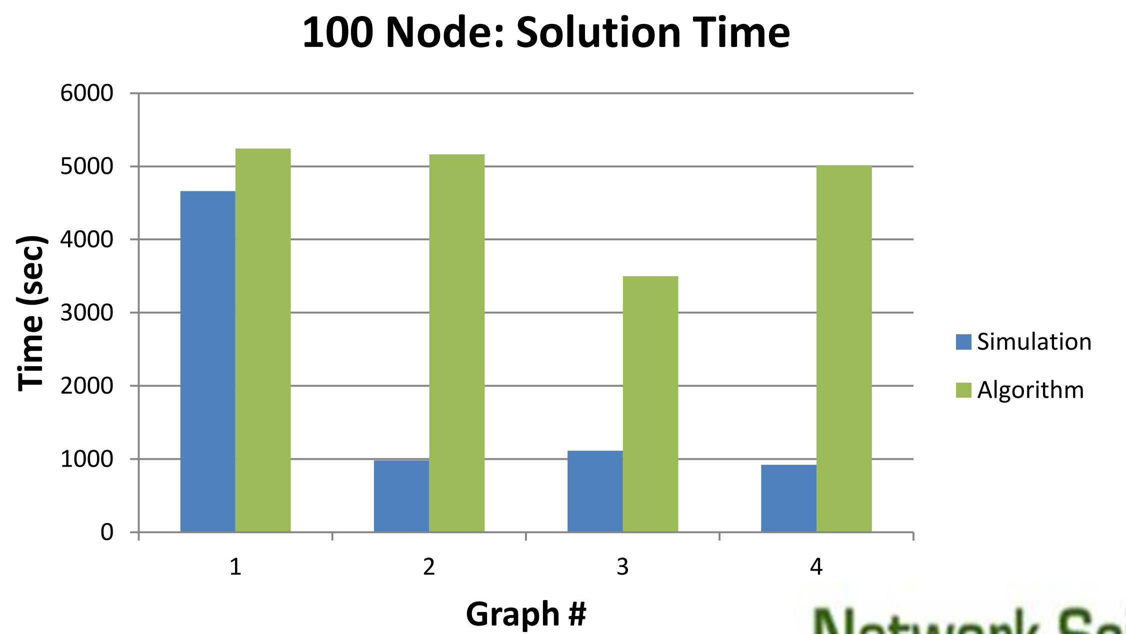

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Future Work

Determine what topographies facilitate a tight lower bound for the algorithm.

Game theory extensions on cooperation mean time to fixation.

References

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